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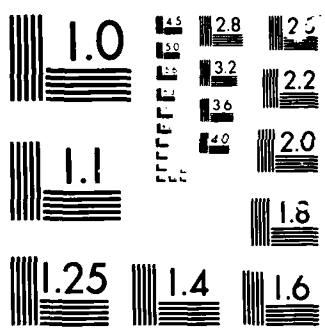
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HAMILTON-JACOBI EQUATIONS IN INFINITE DIMENSIONS, PART III

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ABSTRACT

This paper is concerned with a number of topics in the theory of viscosity solutions of Hamilton-Jacobi equations in infinite dimensional spaces begun in Parts I and II of this series. The development of the theory in the generality in which the "space" or state variable lies in an infinite dimensional space is partly motivated by the hope of eventual applications to the theory of control of partial differential equations or control under partial observation. Among the results presented are: The existence and uniqueness theory previously discussed in spaces with the Radon-Nikodym property is extended beyond this class; examples are given which show that Galerkin approximation arguments in their naive forms cannot be made the basis of an existence theory; some equations with "unbounded terms" of the sort that arise in control of pde's are treated by means of a change of variables reducing the problem to the previously studied cases.

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HAMILTON-JACOBI EQUATIONS IN INFINITE DIMENSIONS, PART III

Michael G. Crandall* and Pierre-Louis Lions**

This paper continues the study of Hamilton-Jacobi equations in infinite dimensions begun in [8] - [10]. The equations of interest have the form

$$F(x,u,Du) = 0 \text{ in } \Omega \tag{HJ}$$

where Ω is an open subset of some real Banach space V, the unknown function $u:\Omega \to R$ is to be continuous, $Du(x)$ denotes the Fréchet derivative of u at $x \in \Omega$ (and thus takes its values in the dual V^* of V), while the nonlinear function F is a continuous mapping from $\Omega \times R \times V^*$ into R (i.e. $F \in C(\Omega \times R \times V^*)$).

In Part I ([9]), we showed that general uniqueness results hold for Hamilton-Jacobi equations in infinite dimensional spaces for the same type of generalized solutions, the so-called viscosity solutions, for which uniqueness was proved in the "classical case" $V = R^N$ in [7] (see also M. G. Crandall, L. C. Evans and P. L. Lions [5]). Corresponding existence results were proved in Part II ([10]) under essentially the same assumptions on the equation and V as used for uniqueness. Various counterexamples showing the necessity of the assumptions on the equation were also given in [9], [10].

A basic assumption made in Parts I and II was that V has the Radon-Nikodym property (i.e., "V is RN"). The Radon-Nikodym property was used everywhere in the analysis in the guise of a result of C. Stegall [23] asserting that if V is RN, then a bounded, continuous real-valued function on a closed ball in V can be perturbed by an

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arbitrarily small linear functional to obtain a function which attains its minimum value. This result was the device which enabled us to overcome the lack of compactness of closed and bounded subsets of V in extending the uniqueness arguments from the classical case. It was also assumed (although the assumption was buried in conditions imposed on the Hamiltonian) that V admits a norm-like function which is differentiable on $V \setminus \{0\}$. Two other new ingredients (beyond the use of Stegall's result mentioned above) were required to treat the existence question; the use of ad hoc differential games to provide existence for "regularized" Hamiltonians and a sharp constructive convergence result [10, Theorem 2.1], which was new even in the classical case.

The topics taken up in the current paper are wide ranging; they include relaxing the requirement that V be RN in the uniqueness and existence theory, existence and uniqueness results when H satisfies coercivity conditions, examples showing the dramatic failure of the basic Fadeo-Galerkin method and the treatment of (SP) and (CP) in certain cases in which the Hamiltonian is not everywhere defined but involves expressions like Ax in which -A is an unbounded generator of a strongly continuous semigroup of contractions. A more detailed description follows.

Section I of this paper is devoted to relaxing the requirement that V be RN in parts of the theory. This is done by introducing the notion of strict viscosity solutions (which was briefly mentioned in [9, Appendix]). It is shown that the notion is stable, coincides with the standard notion in nice spaces and that if (for example) the norm of V is differentiable except at the origin, then general uniqueness results hold for both the model stationary problem

(SP) $$u + H(x,u,Du) = 0 \text{ in } V$$

and the Cauchy problem

(CP) $$u_t + H(x,t,u,Du) = 0 \text{ in } V \times (0,T]$$
$$u(x,0) = \varphi(x).$$

The principal new technical aspect involves using the notion of strict viscosity solutions to work with Ekeland's principle [13] (which holds in general Banach spaces) in place of Stegall's result. However, this theory still requires V to satisfy some version of the condition:

(0) There is a mapping $N: V \to [0,\infty[$ which is Lipschitz continuous on V, differentiable on $V \backslash \{0\}$ and satisfies $N(x) \geq |x|$ for $x \in V$.

We note that this condition excludes extremely important choices for V like L^1, L^∞, spaces of continuous functions and the space of bounded measures. The study of (HJ) equations in these spaces remains almost totally open. After the discussion of uniqueness, we present complementary existence results for (SP) and (CP). In the discussions of both the existence and uniqueness results we give very little detail and merely refer the reader to the program of Parts I and II when it is easily adapted to the current setting. When essentially new arguments are required, we give at least one example. In particular, the basic scheme of the existence proof is that which was introduced in Part II ([10]). However, the method used in [10] for viscosity solutions breaks down for strict viscosity solutions in the final stages of the argument and this new difficulty is overcome by considering two distinct differential games for the same equation.

Section II is devoted to three distinct topics. First, it is shown how to obtain existence and uniqueness results when the Hamiltonian in (S) or (CP) is coercive; that is $H(x,p) \to \infty$ as the norm of $p \in V^*$ tends to ∞. Secondly, we give examples which show the dramatic failure of the Faedo - Galerkin approximation method (in a simple form) as a vehicle to prove the existence results. Indeed, natural finite dimensional approximations may converge to a function solving the wrong equation! Finally, we consider some Hamilton - Jacobi equations associated with control problems for evolution equations. These have the form

(1) $$u_t + (Ax,Du) + F(t,Du) = 0 \text{ in } V \times]0,T[$$

where V is a Hilbert space and the (possibly unbounded) linear operator $-A$ is the infinitesimal generator of a strongly continuous semigroup on V. Similar equations were first studied by V. Barbu and G. DaPrato [1], [2]. The term (Ax,p) occurring in the Hamiltonian is only defined if x lies in the domain of A, so (1) is not covered by the results of Parts I and II. However, we introduce a device which allows one to reduce (1) to a form where the arguments of Parts I and II still succeed. As the expert reader will have noticed, the form of (1) is not yet general enough to cover many applications to the control of pde's one might hope to encompass in the theory, and significant extensions of (1) are under investigation.

We would like to mention the fact that it is typical of the subject that all existence and uniqueness results of the sort we present here have many variations. For example, the following problems have been studied in the case $V = \mathbb{R}^N$ and the results may be extended to the infinite dimensional setting: Problems with boundary conditions of either Dirichlet type (M. G. Crandall and P. L. Lions [7], P. L. Lions [18], [19], G. Barles [3], [4]) or Neumann type (P. L. Lions [20], B. Perthame and R. Sanders [22]) and problems whose solutions have different behaviours at ∞ (H. Ishiii [16] and M. G. Crandall and P. L. Lions [12]).

We have assumed that potential readers (s?) of this paper have had significant experience with viscosity solutions and Parts I and II are a prerequisite to following the text. Indeed, while the statements of the results will be clear enough, the assumptions themselves will be totally unpalatable without prior exposure.

I. STRICT VISCOSITY SOLUTIONS

I.1. Definition, Elementary Properties and Uniqueness Results.

We begin by recalling the definitions of ε-super and subdifferentials of $u \in C(\Omega)$ at $x \in \Omega$ where Ω is an open subset of the Banach space V. First let us fix some notation. We will use $|\ |$ to denote the norm in V, the dual norm in V^* and the absolute value on $\mathbb{R}$. B_R (B_R^*) will denote the ball of radius R and center 0 in V (respectively, V^*) while $B(x,r)$ ($B^*(p,r)$) is the ball of center x (respectively, p) and radius r in V (respectively, V^*). We will not distinguish between open and closed balls; the reader can deduce which is appropriate from the context. The value of $p \in V^*$ at $x \in V$ will be denoted by (p,x).

The ε - superdifferential of φ at x, $D_\varepsilon^+ u(x)$, is given by

$$(2) \qquad D_\varepsilon^+ u(x) = \{p \in V^*: \limsup_{\substack{y \to x \\ y \in \Omega}} \left(\frac{u(y) - u(x) - (p, y - x)}{|y - x|} \right) < \varepsilon\}$$

and, similarly, the ε - subdifferential $D_\varepsilon^- u(x)$ is given by

$$(3) \qquad D_\varepsilon^- u(x) = \{p \in V^*: \limsup_{\substack{y \to x \\ y \in \Omega}} \left(\frac{u(y) - u(x) - (p, y - x)}{|y - x|} \right) > -\varepsilon\} .$$

Here ε is any positive number. A little thought shows that another way to say that $p \in D_\varepsilon^+ u(x)$ (respectively, $p \in D_\varepsilon^- u(x)$) is to say that there is a $\psi \in C(\Omega)$ which is differentiable at x with the derivative $D\psi(x) = p$ and a number δ, $0 \le \delta < \varepsilon$ such that $u(y) - \psi(y) - \delta|x - y|$ (respectively, $u(y) - \psi(y) + \delta|x - y|$) has a local maximum (respectively, minimum) with respect to y at $y = x$.

Now we define strict viscosity sub and supersolutions. Let $F \in C(\Omega \times \mathbb{R} \times V^*)$.

Definition I.1: If $\gamma > 0$ and $u \in C(\Omega)$, then u is a strict viscosity subsolution up to γ of $F = 0$ in Ω if

$$(4) \qquad \inf_{|q| \le \varepsilon} F(x,u(x),p + q) \le 0 \quad \text{for } x \in \Omega,\ p \in D_\varepsilon^+ u(x) \text{ and } 0 < \varepsilon < \gamma$$

and u is a strict viscosity supersolution up to γ of $F = 0$ in Ω if

$$(4)' \qquad \sup_{|q| \le \varepsilon} F(x,u(x),p + q) \ge 0 \text{ for } x \in \Omega,\ p \in D_\varepsilon^- u(x) \text{ and } 0 < \varepsilon < \gamma.$$

Of course, u is a strict viscosity solution of $F = 0$ up to γ if it is both a strict viscosity subsolution and a strict viscosity supersolution up to γ. If u is a strict viscosity subsolution (supersolution, solution) up to γ for all $\gamma > 0$, then it is simply a strict viscosity subsolution (respectively, supersolution, solution). Finally, a strict viscosity subsolution (respectively, supersolution, solution) of $F = 0$ will be referred to as an strict viscosity solution of $F \leq 0$ (respectively, $F \geq 0$, $F = 0$).

Remarks I.1:

(i) The usual sub and superdifferentials $D^+u(x)$ and $D^-u(x)$ are related to the ε-versions by $D^{\mp}u(x) = \bigcap_{\varepsilon>0} D_\varepsilon^{\mp}u(x)$. It then follows from the continuity of F and letting ε tend to zero in (4) and (4)' that strict viscosity sub and supersolutions are viscosity sub and supersolutions in the ordinary sense. As usual, (4) is regarded as satisfied automatically at points x for which $D_\varepsilon^+u(x)$ is empty, etc.

(ii) We have changed the definition slightly from that given (as an example among various possibilities) in Part I ([9]): In [9] we used (4) and (4)' but with

$$\overline{D}_\varepsilon^\nu u(x) = \bigcap_{\eta>\varepsilon} D_\eta^\nu u(x) \text{ for } \varepsilon > 0 \text{ and } \nu \in \{+,-\}$$

in place of $D_\varepsilon^\nu u(x)$. Since these sets are larger than those given by (2) and (3), the notion defined in [9] is stronger than that given here. If, for example, $F(x,r,\cdot) \in BUC(B_R^*)$ for $R > 0$, $x \in \Omega$ and $r \in \mathbb{R}$ then the notions are equivalent.

(iii) The definition of strict viscosity sub and supersolutions apparently depends on the choice of norms in V and V^*. We are taking the norm in V^* to be the dual of the norm on V. To obtain a notion invariant under changing the norm of V or V^* to an equivalent norm one would have to allow $|q| \leq C\varepsilon$ in (4) and (4)'.

(iv) Many variants are possible: One may replace $|q| \leq \varepsilon$ by any ε-neighborhood of the origin containing the ball B_ε^* and even allow this neighborhood to depend on x. One could also require (4) and (4)' to hold simultaneously for all equivalent norms on V with the corresponding dual norms on V^*. One may still prove uniqueness using many of these various notions.

(v) If u is differentiable at x, then $D_\varepsilon^{\mp}u(x) = B^*(Du(x),\varepsilon)$. Hence if $F(x,u(x),Du(x)) = 0$, then (4) and (4)' also hold at x.

To begin the theoretical development, we will prove that the basic stability property of classical viscosity solutions is enjoyed by strict viscosity solutions; that is this class of solutions is stable (closed) with respect to the topology of local uniform convergence. In fact, we prove a little more below. Here and later, expressions like $a_n \to a$ mean $\lim_{n\to\infty} a_n = a$. We will say a sequence of functions f_n on a subset Ω of a metric space converges <u>locally uniformly</u> (or $f_n \to f$ locally uniformly) on Ω to a limit f if each point $x \in \Omega$ has a neighborhood O such that $f_n(y) \to f(y)$ uniformly for $y \in O$ and that f_n <u>converges continuously</u> to f (or $f_n \to f$ continuously) on Ω if whenever $\Omega \ni x_n \to x \in \Omega$, then $f_n(x_n) \to f(x)$. In the event that the functions f_n also depend on some parameters $\lambda \in \Lambda$, it is clear what one means by $f_n \to f$ locally uniformly or continuously, uniformly for $\lambda \in \Lambda$.

While continuous convergence to a continuous limit is weaker than local uniform convergence, we do not have any applications in mind at the moment for the added generality provided by formulating one of the hypotheses in the following proposition in terms of continuous convergence.

<u>Proposition I.1</u>: Let $\gamma > 0$, u_n, $u \in C(\Omega)$ and F_n, $F \in C(\Omega \times \mathbb{R} \times V^*)$ for $n = 1,2,\ldots$. Let u_n be a strict viscosity solution of $F_n \le 0$ (respectively, $F_n \ge 0$) up to γ in Ω and assume that

(5) $\quad u_n \to u$ locally uniformly and $F_n(\cdot,\cdot,p) \to F_\infty(\cdot,\cdot,p)$ continuously

on $\Omega \times \mathbb{R}$ uniformly in $p \in B_R^*$ for $R > 0$. Then u is a strict viscosity solution of $F \le 0$ respectively, $F \ge 0$) up to γ in Ω.

<u>Proof</u>: We treat the subsolution case. Let $\varepsilon > 0$, $x \in \Omega$ and $p \in D_\varepsilon^+u(x)$. According to the definitions and asumptions there will then exist an $r > 0$ and

$\bar{\varepsilon} \in (0,\varepsilon)$ such that

(6) $$u(y) - (p,y - x) - \bar{\varepsilon}|y - x| \le u(x) \text{ for } y \in B(x,r)$$

and $u_n \to u$ uniformly on $B(x,r)$. Choose $\eta \in (\bar{\varepsilon},\varepsilon)$ and set

$$\varphi(y) = u(y) - (p,y - x) - \eta|y - x|,$$

$$\varphi_n(y) = u_n(y) - (p,y - x) - \eta|y - x|,$$

for $y \in B(x,r)$. Put

(7) $$\delta_n = \sup_{B(x,r)} |u_n - u|$$

and, assuming that $\delta_n > 0$ (the other case being trivial), choose $x_n \in B(x,r)$ so that

(8) $$\varphi_n(x_n) > \sup_{B(x,r)} (\varphi_n - \delta_n) .$$

It follows from Ekeland's theorem [13] that there is a $y_n \in B(x,r)$ such that

(9) $$\varphi_n(y_n) \ge \varphi_n(x_n) \ge \sup_{B(x,r)} (\varphi_n - \delta_n) \text{ and}$$

$$\varphi_n(y) - \delta_n|y - y_n| \le \varphi_n(y_n) \text{ for } y \in B(x,r).$$

In view of (6), (7) we see that

(10) $$\varphi_n(y_n) \le \varphi(y_n) + \delta_n \le \varphi(x) + \delta_n - (\eta - \bar{\varepsilon})|y_n - x|$$

while

(11) $$\varphi_n(y_n) \ge \varphi_n(x_n) \ge \varphi_n(x) - \delta_n \ge \varphi(x) - 2\delta_n.$$

Hence $|y_n - x| \le 3(\bar{\varepsilon} - \eta)^{-1}\delta_n$ and so $y_n \to x$ as $n \to \infty$. In particular, y_n is in the interior of $B(x,r)$ for large n and it follows from the second part of (9) that

$$u_n(y) - (p,y - x) - (\eta + \delta_n)|y - y_n| \le u_n(y_n) - (p,y_n - x)$$

for $y \in B(x,r)$. Thus $p \in D_\varepsilon^+u_n(y_n)$ for n large enough and

$$\inf_{|q|\le\varepsilon} F_n(y_n,u_n(y_n),p + q) \le 0 \text{ for } \varepsilon < \gamma.$$

The result follows upon invoking the assumption (5).

We remark that slight changes in the proof above allow us to weaken the assumption that $u_n \to u$ locally uniformly. Indeed, in the case of subsolutions (supersolutions) it is enough to have $(u_n - u)^+ \to 0$ (respectively, $(u - u_n)^+ \to 0$) locally uniformly and $u_n \to u$ pointwise.

We next observe that viscosity solutions are strict viscosity solutions if the norm of V is differentiable off the origin and V is RN.

Proposition I.2: Let V be RN and the norm of V be differentiable off the origin. Let $F \in C(\Omega \times \mathbb{R} \times V^*)$ and assume that if $x_n \to x$ and $r_n \to r$, then $F(x_n,r_n,p) \to F(x,t,p)$ uniformly for $p \in B_R^*$ for $R > 0$. If $u \in C(\Omega)$ is a viscosity subsolution (supersolution) of $F = 0$ on Ω, then it is an strict viscosity solution of $F \leq 0$ (respectively, $F \geq 0$) in Ω.

In fact, this was shown in the appendix of [9] if V is finite dimensional with $\overrightarrow{D_\varepsilon}$ in place of $D_\varepsilon^{\mp}$ (see Remarks I.1(ii) above). The proof given in [9] adapts in a straightforward way to the current case.

Our next task is to present the uniqueness results. By contrast with Part I, we will no longer assume that V is RN. However, we will still require the existence of differentiable norm-like functions on V so that, in particular, Proposition I.1 holds. A well-known example of a space which is not RN but which admits such functions is provided by the space c_0 of sequences which converge to 0.

Example: The space c_0 of sequences $\{a_n\}_{n\geq 1}$ satisfying $a_n \to 0$ has a norm which is infinitely differentiable on the complement of the origin - see, for example, M. Leduc [17] - but it is not RN. Indeed, to see that the form of the Radon-Nikodym property of greatest interest to us fails, the reader may easily check that the function

$$\varphi(\{a_n\}_{n\geq 1}) = \sup_{n\geq 1} (1 - 1/n)|a_n|$$

has the property that no perturbation of it by a continuous linear functional attains its maximum value on the closed unit ball. Thus, by Stegall's result [23], c_0 is not RN.

The uniqueness results will be formulated in the context of the model problems (SP) and (CP). The rather unpleasant task of formulating the conditions on the

Hamiltonian H in these problems is taken up next. As in Part II, we will state conditions on $H(x,t,r,p)$ where $H:V \times \mathbf{R} \times [0,T] \times V^* \to \mathbf{R}$ and interpret these in the context of (SP) as applied to a t-independent Hamiltonian $H(x,r,p)$.

We will make the following hypothesis throughout the paper and simplify the presentation by nevermore referring to it:

<u>THE BLANKET CONTINUITY ASSUMPTION</u>: H is bounded and uniformly continuous on $B_R \times [0,T] \times [-R,R] \times B_R^*$ for all $R > 0$.

In the statements below, a modulus is a continuous nondecreasing and subadditive mapping $m:[0,\infty) \to [0,\infty)$ with $m(0) = 0$ and a local modulus is a continuous mapping $\sigma:[0,\infty) \times [0,\infty) \to [0,\infty)$ nondecreasing in both variables such that $r \to \sigma(r,R)$ is a modulus for each $R > 0$. The assumptions we will use on H involve the existence of certain auxiliary functions

$$d:V \times V \to \mathbf{R},\ \nu:V \to \mathbf{R} \text{ and } \mu:V \to \mathbf{R}$$

which satisfy the conditions (C) below and with respect to which H will be required to satisfy combinations of the conditions below:

(H1) For all $R > 0$ there is a constant C_R such that

$$H(x,t,r,p) - H(x,t,r,p + \lambda D\nu(x)) \le C_R$$

for $x \in V$, $t \in [0,T]$, $r \in \mathbf{R}$, $p \in B_R^*$, $0 \le \lambda \le R$.

(H2) There is a local modulus σ such that

$$H(x,t,r,p) - H(x,t,r,p + \lambda D\mu(x)) \le \sigma(\lambda,|p|)$$

for $x \in V$, $t \in [0,T]$, $r \in \mathbf{R}$, $p \in V^*$, $0 \le \lambda \le 1$.

(H3) There is a modulus m such that

$$H(y,t,r,-\lambda d_y(x,y)) - H(x,t,r,\lambda d_x(x,y)) \le m(\lambda d(x,y) + d(x,y))$$

for $x \ne y \in V$, $t \in [0,T]$, $r \in \mathbf{R}$, $p \in V^*$, and $0 \le \lambda$ and $d(x,y) \le 1$.

(H3w) There is a local modulus κ such that

$$H(y,t,r,-\lambda d_y(x,y)) - H(x,t,r,\lambda d_x(x,y)) \le \kappa(d(x,y),\lambda)$$

for $x \neq y \in V$, $t \in [0,T]$, $r \in R$, $p \in V^*$ and $0 \le \lambda$.

(H4) $H(x,t,r,p)$ is nondecreasing with respect to r for $(x,t,p) \in V \times [0,T] \times V^*$.

The functions d, μ, ν satisfy the following conditions which we collectively denote by (C).

(C) d, μ, ν are Lipschitz continuous, $d(x,y)$ is differentiable in x for $x \neq y$ and in y for $y \neq x$, ν, μ are nonnegative and differentiable on V, $\mu(x) \to \infty$ as $|x| \to \infty$, $\nu(x) \ge |x|$ for large $|x|$, $d(x,y) \ge |x - y|$ on $V \times V$, and $d(x,x) = 0$ on V.

We observe that (C) implies (0), for if d has the properties of (C), then setting $N(x) = d(x,0)$ yields a function satisfying the requirements of (0).

<u>Theorem I.1 (Uniqueness for (CP))</u>: Let (H2), (H3), (H4) and (C) hold and $\gamma > 0$. Let $f \in C_b(V \times [0,T])$ and u, $v \in BUC(B_R \times [0,T])$ for all $R > 0$. Let u and v be, respectively, a strict viscosity solution up to γ of

$$u_t + H(x,t,u,Du) \le 0 \text{ and } v_t + H(x,t,v,Dv) + f(x,t) \ge 0$$

in $V \times]0,T[$. Let u and v be uniformly continuous in x uniformly for $t \in [0,T]$. Assume also that either (H1) holds or that $u - v$ is bounded from above. Then

$$u(x,t) - v(x,t) \le \sup_{y \in V} (u(y,0) - v(y,0))^+ + \int_0^t \sup_{y \in V} f(y,s)^+ ds \quad . \tag{12}$$

Finally, if u or v is Lipschitz continuous on V uniformly in $t \in [0,T]$, then (12) still holds if (H3) is weakened to (H3w).

<u>Theorem I.2 (Uniqueness for (SP))</u>: Let (H2), (H3), (H4) and (C) hold. Let $f \in C_b(V)$. Let u, $v \in UC(V)$ and be, respectively, a strict viscosity solution of $u + H(x,u,Du) \le 0$ and $v + H(x,v,Dv) + f(x) \ge 0$ up to γ. Assume also that either (H1) holds or $u - v$ is bounded from above. Then

$$u(x) - v(x) \le \sup_{y \in V} f(y)^+ \quad \text{for } x \in V . \tag{13}$$

Finally, if u or v is Lipschitz continuous on V, then (13) still holds if (H3) is weakened to (H3w).

Remarks I.2:

(i) The reader will find some discussion concerning the formulation of the hypotheses (H1) - (H4) in Parts I and II as well as examples illustrating the scope and necessity of various conditions.

(ii) (H3) may be generalized as in M. G. Crandall and P. L. Lions [9].

(iii) If (0) holds and H satisfies

$$H \in UC(V \times [0,T] \times \mathbf{R} \times B_R^*) \tag{14}$$

for all $R > 0$ and there is a modulus m such that

$$|H(x,t,r,p) - H(y,t,r,p)| \le m(|x - y|(1 + |p|)) \tag{15}$$

for $x,y \in V$, $t \in [0,T]$, $r \in \mathbf{R}$ and $p \in V^*$, then (H1), (H2) and (H3) hold with $d(x,y) = N(x - y)$, ν, $\mu = (1 + N^2)^{1/2}$.

(iv) A simple but striking and useful extension of the above results comes about as follows: In the case of Theorem I.2, assume that there is a nondecreasing function $g(\varepsilon) > 0$ such that $g(0+) = 0$ and rather than being a strict viscosity solution of $u + H(x,u,Du) \le 0$ in V, $u \in UC(V)$ satisfies

$$u(x) + \inf_{|q| \le \varepsilon} H(x,u(x),p + q) \le g(\varepsilon) \text{ for } x \in V,\ p \in D_\varepsilon^+ u(x) \text{ and } 0 < \varepsilon \le \gamma ,$$

while v satisfies the corresponding weakened notion of strict viscosity solution of $v + H(x,v,Dv) + f \ge 0$. Then the assertions of Theorem I.2 remain correct. A similar remark holds for Theorem I.1. Indeed, in the case of Theorem I.2, one just observes that for $\varepsilon \le \gamma$ u is a strict viscosity solution of $u + H(x,u,Du) \le g(\varepsilon)$ up to ε, etc., applies the previous result and lets $\varepsilon \downarrow 0$.

We will illustrate the main new ingredient in the proof of these theorems, which is the use of Ekeland's principle at the point in the argument where Stegall's result was previously used, in the simplest case. That is, we will assume $u, v \in BUC(V)$ are strict viscosity solutions of

(16) $$u + H(Du) \le 0 \text{ and } v + H(Dv) \ge 0$$

where $H \in UC(B_R)$ for all $R > 0$. Assuming, moreover, that (0) holds puts us in the situation of Remarks 2(iii) with $m \equiv 0$. We follow the arguments of Part I with the change just remarked on and modified to reflect the point of view of M. G. Crandall, H. Ishii and P.-L. Lions [6]. Let $Z = V \times V$ and for $z = [x,y] \in Z$

$$w(z) = u(x) - v(y).$$

Then it is straightforward to check that w is an strict viscosity solution of

(18) $$w + \hat{H}(Dw) \le 0 \text{ on } Z$$

where $\hat{H} \in C(V^* \times V^*)$ is given by

$$\hat{H}([p,q]) = H(p) - H(q).$$

Define Φ by $\Phi([x,y]) = BN(x - y)^2$ where B is a parameter to be chosen. Then

(19) $$\Phi(z) + \hat{H}(D\Phi(z)) \ge 0$$

pointwise in Z (and so Φ is a strict viscosity solution of the same inequality). (Of course, (19) holds trivially here since $\hat{H}(D\Phi(z)) \equiv 0$, but in more general cases one constructs corresponding differentiable supersolutions.) We want to prove that $u \le v$. It will suffice to show that if B is large, then

$$w([x,y]) = u(x) - v(y) \le \Phi([x,y]) \text{ on } \Omega = \{[x,y]: |x - y| < 1\}$$

since $\Phi([x,x]) \equiv 0$. Let η, $\delta > 0$ and $\bar{z} \in \bar{\Omega}$ satisfy

(20) $$\begin{aligned} &w(\bar{z}) - \Phi(\bar{z}) \ge \sup_{\Omega} (w - \Phi) - \delta \text{ and} \\ &w(z) - \Phi(z) - \eta|z - \bar{z}| \le w(\bar{z}) - \Phi(\bar{z}) \quad \text{for } z \in \Omega. \end{aligned}$$

The existence of $\bar{z}$ is from Ekeland's theorem. Using the boundedness of u and v, one sees that if B is large and δ small, then $\bar{z} \in \Omega$. The second inequality then implies that $D\Phi(\bar{z}) \in D_\varepsilon^+ w(\bar{z})$ for $\varepsilon > \eta$. Hence, (19) in the pointwise sense and (18) in the strict viscosity solution yields

$$w(\bar{z}) - \Phi(\bar{z}) < w(\bar{z}) - \hat{H}(D\Phi(\bar{z})) < w(\bar{z}) + \inf_{|p|,|q|<\varepsilon} \hat{H}(D\Phi(\bar{z}) + [p,q]) +$$

$$\sup_{|p|,|q|<\varepsilon} \left|\hat{H}(D\Phi(\bar{z}) + [p,q]) - \hat{H}(D\Phi(\bar{z}))\right| < \sup_{|p|,|q|<\varepsilon} \left|\hat{H}(D\Phi(\bar{z})) + [p,q]) - \hat{H}(D\Phi(\bar{z}))\right|.$$

In conjunction with (20) we thus have

$$\sup_{\Omega} (w - \Phi) < \sup_{|p|,|q|<\varepsilon} \left|\hat{H}(D\Phi(z) + [p,q]) - \hat{H}(D\Phi(z))\right| + \delta$$

Since Φ is Lipschitz continuous, $\hat{H}$ is uniformly continuous on bounded sets and η, δ and $\varepsilon > \eta$ are arbitrary, the result follows upon sending ε and δ to 0.

I.2. Existence and Relations With Differential Games.

We begin with an existence result for the Cauchy problem (CP).

Theorem I.3: Let $T > 0$, $\varphi \in UC(V)$. Let H satisfy (H2), (H3) and (H4).

(i) If (H1) holds, then there exists a unique strict viscosity solution u of (CP) such that $u \in BUC(B_R \times [0,T])$ for $R > 0$ and u is uniformly continuous in V uniformly for $t \in [0,T]$.

(ii) If $\varphi \in BUC(V)$, $H(x,t,0,0) \in C_b(V \times [0,T])$, then there exists a unique bounded strict viscosity solution u of (CP) such that $u \in BUC(B_R \times [0,T])$ for $R > 0$, and u is uniformly continuous in $x \in V$ uniformly for $t \in [0,T]$.

Existence for the stationary problem requires some additional assumptions on H. For example:

(H5) There is a function $F:[0,\infty) \times [0,\infty) \to [0,\infty)$ which is nondecreasing in its arguments such that

$$H(y,r,-\lambda d_y(x,y)) - H(x,r,\lambda d_x(x,y)) \leq F(\lambda,d(x,y))$$

for $x, y \in V$, $r \in R$, $\lambda > 0$ and a nondecreasing uniformly continuous map $G:[0,\infty) \to R$ which is continuously differentiable on $(0,\infty)$ and satisfies $G(r) > F(G'(r),r)$ on $(0,\infty)$.

Theorem I.4: Let H satisfy (H2), (H3) and (H4).

(i) If (H1), (H5) hold there is a unique strict viscosity solution of (SP) in UC(V).

(ii) If $H(x,0,0) \in BUC(V)$, there exists a unique strict viscosity solution of (SP) in BUC(V).

Sketch of proof of theorems I.3 and I.4: Again, the outline of the existence proof in Part II can be followed through Sections 2 and 3 of [10]. However, the arguments of Section 4 must be modified substantially to accommodate the strict viscosity notion. We will explain the required modifications in the following model case:

(20) $$u + H(x,Du) = 0 \text{ in } V$$

where H is bounded and Lipschitz on $V \times V^*$. One can still formulate differential games whose value functions are viscosity solutions of (20) (as was shown in [10]), but we do not know how to show directly that they are strict viscosity solutions of (20) - indeed, depending on which game we choose we will only be able to show that the value function satisfies either

$$
(21) \qquad
\begin{aligned}
&u(x) + \inf_{|q|\le\varepsilon} H(x,p+q) \le L\varepsilon \text{ for } x\in V \text{ and } p \in D_\varepsilon^+ u(x),\\
&u(x) + \sup_{|q|\le\varepsilon} H(x,p+q) \ge 0 \quad \text{for } x \in V \text{ and } p\in D_\varepsilon^- u(x) .
\end{aligned}
$$

or

$$
(22) \qquad
\begin{aligned}
&u(x) + \inf_{|q|\le\varepsilon} H(x,p+q) \le 0 \text{ for } x\in V \text{ and } p \in D_\varepsilon^+ u(x),\\
&u(x) + \sup_{|q|\le\varepsilon} H(x,p+q) \ge - C\varepsilon \quad \text{for } x \in V \text{ and } p\in D_\varepsilon^- u(x)
\end{aligned}
$$

where L is a constant depending on H. However, this is enough, because we can use the concluding remarks of Section 1 to deduce that the two value functions coincide, and thus we have a strict viscosity solution solution of (20).

We now explain how to form the differential games we use. For $R > 0$ put

$$
(23) \qquad
\begin{aligned}
&\bar{H}_R(x,p) = \inf_{q\in B_R^*} \{H(x,q) + L|p-q|\}\\
&\underline{H}_R(x,p) = \sup_{q\in B_R^*} \{H(x,q) - L|p-q|\}
\end{aligned}
$$

where L is a Lipschitz constant for H with respect to p. Clearly

$$
(24) \qquad \bar{H}_R(x,p) \ge H(x,p) \ge \underline{H}_R(x,p) \text{ on } V \times V^* \text{ and } \bar{H}_R = H = \underline{H}_R \text{ on } V \times B_R^*.
$$

Next we observe that we may write $\bar{H}_R$, $\underline{H}_R$ as follows:

$$
(25) \qquad
\begin{aligned}
&\bar{H}_R(x,p) = \inf_{q\in B_R^*} \sup_{z\in B_L} \{ H(x,q) - (p,z) + (q,z)\}\\
&\underline{H}_R(x,p) = \sup_{q\in B_R^*} \inf_{z\in B_L} \{ H(x,q) - (p,z) + (q,z)\}.
\end{aligned}
$$

The developments which follow will produce "value functions" $\bar{u}_R$, $\underline{u}_R \in BUC(V)$ which are Lipschitz continuous with constant L associated with these representations of $\bar{H}_R$ and $\underline{H}_R$

and corresponding differential games. These functions will be shown to satisfy

$$
(26)\qquad
\begin{aligned}
&\bar{u}_R(x) + \inf_{|q|<\varepsilon} \bar{H}_R(x,p+q) < 0 \text{ for } x \in V,\ p \in D^+_\varepsilon u(x),\ \varepsilon > 0,\\
&\bar{u}_R(x) + \sup_{|q|<\varepsilon} \bar{H}_R(x,p+q) > -L\varepsilon \text{ for } x \in V,\ p \in D^-_\varepsilon u(x),\ \varepsilon > 0,
\end{aligned}
$$

and

$$
(27)\qquad
\begin{aligned}
&\underline{u}_R(x) + \inf_{|q|<\varepsilon} \underline{H}_R(x,p+q) < L\varepsilon \text{ for } x \in V,\ p \in D^+_\varepsilon u(x),\ \varepsilon > 0,\\
&\underline{u}_R(x) + \sup_{|q|<\varepsilon} \underline{H}_R(x,p+q) > 0 \text{ for } x \in V,\ p \in D^-_\varepsilon u(x),\ \varepsilon > 0.
\end{aligned}
$$

We would prefer to have H in place of the approximations $\bar{H}_R$, $\underline{H}_R$ in the relations above. To attain this, we need only recall (24), show (by construction) that $\bar{u}_R$ and $\underline{u}_R$ are Lipschitz with constant L and then observe that

$$D^{\mp}_\varepsilon w(x) \subset B^*_{L+\varepsilon}$$

whenever w is Lipschitz with constant L. Putting these things together, the arguments $q + p$ in (26), (27) belong to $B^*_{L+2\varepsilon}$, and thus we may replace $\bar{H}_R$, $\underline{H}_R$ by H in (26), (27) provided $L + 2\varepsilon < R$.

We next build the value functions $\bar{u}_R$, $\underline{u}_R$ which satisfy the above conditions. Consider the following sets of "controls":

$$Q = \{\text{strongly measureable maps } q:[0,\infty) \to B^*_R\}$$
$$Z = \{\text{strongly measurable } z:[0,\infty) \to B_L\}$$

and the corresponding strategies

$$\hat{Z} = \{\text{nonanticipating maps } \hat{z}:Q \to Z\}$$

where a strategy is nonanticipating if whenever two controls agree a.e. on some interval [0,a], then their images by the strategy agree a.e. on the same interval. We set

$$f(x,z,q) = -H(x,q) - (q,z)$$

and for $q \in Q$, $\hat{z} \in \hat{Z}$, and $t > 0$

$$X_t = x + \int_0^t \hat{z}[q](s)ds$$

where we suppress some of the arguments of X in the notation. Then put

$$\bar{u}_R(x) = \inf_{\hat{z}\in\hat{Z}} \sup_{q\in Q} \int_0^\infty f(X_t,\hat{z}[q](t),q(t))e^{-t}dt$$

$$\underline{u}_R(x) = \sup_{\hat{z}\in\hat{Z}} \inf_{q\in Q} \int_0^\infty f(X_t,\hat{z}[q](t),q(t))e^{-t}dt .$$

It is immediate from these formulae that $\bar{u}_R$, $\underline{u}_R$ are Lipschitz continuous on V with constant L. We now have to check (26) and (27). The crucial dynamic programming principle reads

$$\bar{u}_R(x) = \inf_{z\in Z} \sup_{q\in Q} \left\{ \int_0^h f(X_t,\hat{z}[q](t),q(t))e^{-t}dt + \bar{u}_R(X_h)e^{-h}\right\}$$

and we immediately deduce from it that

$$\sup_{\hat{z}\in\hat{Z}} \inf_{q\in Q} \left(\frac{1}{h}(\bar{u}_R(x) - \bar{u}_R(X_h)) - \frac{1}{h}\int^h f(x,\hat{z}[q](t),q(t))dt\right) + \bar{u}_R(x) = \gamma(h)$$

where $\gamma(h) \to 0$ as $h \to 0$.

We may now check the second inequality in (26) very easily. For notational simplicity we now set $w = \bar{u}_R$. Let $\varepsilon > 0$, $x \in V$, and $p \in D_\varepsilon^- w(x)$. Then there are r, h_0 > 0 such that

$$w(y) \geq w(x) + (q,y - x) - \varepsilon|y - x| \text{ for } |y - x| < r,$$

and

$$|X_h - x| < r \text{ for all } \hat{z} \in \hat{Z} \text{ and } q \in Q \text{ if } h < h_0.$$

The above dynamic programming equality then implies that for $h < h_0$ and any fixed $q \in B_R^*$ (regarded as a constant control), we have

$$\sup \left((q - p,\frac{1}{h}\int_0^h z[q](t)) + \varepsilon\left|\frac{1}{h}\int_0^h \hat{z}[q](t)dt\right|\right) + H(x,q) + w(x) \geq - \gamma(h).$$

This implies, upon letting h tend to 0, that

$$L|q - p| + L\varepsilon + H(x,q) + w(x) > 0,$$

and since $q \in B_R^*$ is arbitrary, we are done.

We prove the first inequality of (26) arguing by contradiction. Let $\varepsilon > 0$, $x \in V$, $p \in D_\varepsilon^+ w(x)$ and

$$\inf_{|\eta|<\varepsilon} \bar{H}_R(x,p + \eta) + w(x) > 4\gamma > 0.$$

This implies that

$$\inf_{q\in B_R} \inf_{|\eta|<\varepsilon} \{ L|p + \eta - q| + H(x,q)\} + w(x) > \gamma$$

or

$$\inf_{q\in B_R} \sup_{z\in B_L} \{ -(p - q,z) - \varepsilon|z| + H(x,q)\} + w(x) > 4\gamma.$$

We can then choose for each $q \in B_R^*$ some $z(q) \in B_L$ (in a strongly measurable way) such that

$$-(p - q,z(q)) - \varepsilon|z(q)| + H(x,q) + w(x) > 3\gamma.$$

Therefore, for each control $q \in Q$ we have

$$-(p - q,z(q(t))) - \varepsilon|z(q(t))| + H(x,q(t)) + w(x) > 3\gamma \text{ a.e.}$$

Defining the strategy $\hat{z}$ by $\hat{z}[q](t) = z(q(t))$, we integrate this inequality over the interval $0 < t < h$ to find

$$\sup_{q\in Q} \{-\frac{1}{h}(p - q(t),X_h - x) - \varepsilon \frac{1}{h}\int_0^h |\hat{z}[q](t)|dt - \frac{1}{h}\int_0^h f(x,\hat{z}[q](t),q(t))dt\}$$

$$+ w(x) > 3\gamma .$$

Since

$$\frac{1}{h}\int_0^h |\hat{z}(q)(t)|dt > |\frac{1}{h}\int_0^h \hat{z}[q](t)dt| = |\frac{1}{h}(X_h - x)|$$

we finally conclude that

$$\inf_{z\in Z} \sup_{q\in Q} \{ \frac{1}{h}(p,x - X_h) - \varepsilon\frac{1}{h}|x - X_h| + \frac{1}{h}\int_0^h f(x,\hat{z}[q](t),q(t))dt\} + w(x) > 3\gamma.$$

This inequality and the dynamic programming equality are inconsistent, establishing the result. The arguments to establish (27) are entirely parallel, and we end the discussion here.

We now describe the relations between differential games and strict viscosity solutions more generally. To have a simple presentation, we will consider only the case of infinite horizon problems (which correspond to stationary Hamilton-Jacobi equations). Analogous considerations hold for the relation between finite horizon problems and the Cauchy problem for Hamilton-Jacobi equations.

Consider metric spaces A and B and let b and f map $V \times A \times B$ into V and $\mathbb{R}$ respectively. We assume that the maps $(\alpha,\beta) \to b(x,\alpha,\beta)$ and $(\alpha,\beta) \to f(x,\alpha,\beta)$ are continuous on $A \times B$ for each $x \in V$ and the following boundedness and continuity assumptions hold:

(28) $$|f(x,\alpha,\beta) - f(y,\alpha,\beta)| \le \omega(|x - y|) \text{ for } (x,\alpha,\beta) \in V \times A \times B$$

for some modulus ω;

(29) $$|f(0,\alpha,\beta)| + |b(x,\alpha,\beta)| \le C \text{ for } (x,\alpha,\beta) \in V \times A \times B,$$

and

(30) $$|b(x,\alpha,\beta) - b(y,\alpha,\beta)| \le C_0|x - y| \text{ for } x,y \in V,\ \alpha \in A \text{ and } \beta \in B$$

for some constants C and C_0. Differential games are constructed from this type of data and the sets of controls and strategies defined next.

$\mathbf{A}$ and $\mathbf{B}$ will denote the set of controls for the α-player and the β-player respectively; i.e., $\mathbf{A}$ ($\mathbf{B}$) is the set of strongly measurable mappings of $[0,\infty)$ into A (respectively, B). We will denote by $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ the strategies for the α-player and β-player. That is, $\hat{\mathbf{A}}$ is the set of nonanticipating maps from $\mathbf{B}$ to $\mathbf{A}$, etc. (The expert reader will notice that we are using Elliot-Kalton's formulation of differential games - see [14] - in this paper, but this is by no means compulsory.) Fixing $\lambda > 0$, the upper and lower values associated with our game are given by:

(31) $$\underline{u}(x) = \inf_{\hat{\alpha}\in\hat{\mathbf{A}}} \sup_{\beta\in\mathbf{B}} \int_0^\infty f(X_s,\hat{\alpha}[\beta](s),\beta(s))e^{-\lambda s}ds \quad \text{for } x \in V\ ,$$

and

$$(32) \qquad \bar{u}(x) = \sup_{\hat\beta \in \hat B} \inf_{\alpha \in A} \int_0^\infty f(X_s, \alpha(s), \hat\beta[\alpha](s)) e^{-\lambda s} ds \quad \text{for } x \in V$$

where the state process X satsifies

$$X_s = x + \int_0^t b(X_s, \tilde\alpha(s), \tilde\beta(s)) ds \text{ for } t > 0$$

with $\tilde\beta = \beta$, $\tilde\alpha = \hat\alpha[\beta]$ in (31) and $\tilde\alpha = \alpha$, $\tilde\beta = \hat\beta[\alpha]$ in (32). The following result states the relationship between $\bar{u}$, $\underline{u}$ and the Isaac's equations

$$(32) \qquad \lambda u + \underline{H}(x, Du) = 0 \text{ in } V$$

and

$$(33) \qquad \lambda u + \bar{H}(x, Du) = 0 \text{ in } V$$

where

$$\underline{H}(x,p) = \inf_{\beta \in B} \sup_{\alpha \in A} \{-(b(x,\alpha,\beta),p) - f(x,\alpha,\beta)\}$$

and

$$\bar{H}(x,p) = \sup_{\alpha \in A} \inf_{\beta \in B} \{-(b(x,\alpha,\beta),p) - f(x,\alpha,\beta)\}$$

on $V \times V^*$.

Proposition I.3: Assume that (28), (29) and (30) hold. Then $\bar{u}$ and $\underline{u} \in UC(V)$ and $u = \bar{u}$ (respectively, $\underline{u}$) satisfies

$$(34) \qquad \begin{aligned} &\lambda u(x) + \inf_{|q| \le \varepsilon} H(x, p+q) \le C_0 \varepsilon \text{ for } x \in V,\ p \in D_\varepsilon^+ u(x) \text{ and } \varepsilon > 0 \\ &\lambda u(x) + \sup_{|q| \le \varepsilon} H(x, p+q) \ge -C_0 \varepsilon \text{ for } x \in V,\ p \in D_\varepsilon^- u(x) \text{ and } \varepsilon > 0 \end{aligned}$$

with $H = \bar{H}$ (respectively, $H = \underline{H}$). Thus, if V satisfies (0), $\bar{u}$ (respectively, $\underline{u}$) is the unique uniformly continuous strict viscosity solution of (33) (respectively, (32)).

Proof: The verification of (34) is a straightforward adaptation of the arguments of L. C. Evans and P. E. Souganidis [15] which, in turn, were the extension to differential games of a remark of P.-L. Lions [18] when the "ε - terms" are treated as in the discussion of $\bar{u}_R$, $\underline{u}_R$ above. The fact that $\bar{u}$, $\underline{u}$ lie in UC(V) is easily seen from the explicit formulae. Next observe that $\bar{H}$, $\underline{H}$ satisfy (14), (15) so that we may use Theorem I.2 and Remark I.2 (iv) to conclude that if there is a uniformly continuous strict viscosity solution u of (33) (respectively, (32)), then $u = \bar{u}$ (respectively, $u = \underline{u}$). Again, as remarked in Part II [10], (15) implies (H5) and we may invoke the existence Theorem I.4 to conclude the proof.

The above result is not a verification theorem because it uses the existence of strict viscosity solutions. In view of this, the fact that (26) and (27) hold for $\bar{u}_R$ and $\underline{u}_R$ may seem surprising - it is not, because one can prove directly (and thus in any space V), for example, that $\bar{u}$ is a strict viscosity subsolution of (33) if $\{(b(x,\alpha,\beta),f(x,\alpha,\beta)):\alpha \in A\}$ is convex in $V \times \mathbf{R}$ for every $x \in V$ and $\beta \in B$. In fact, by considering generalized controls and/or strategies (as in P.L. Lions and P. E. Souganidis [20]), this may always be achieved.

We conclude this section by showing that for control problems the situation is slightly better. For purposes of illustration, we take the particular case in the above set up in which $B = \phi$ and b, f depend only on x and α. Thus we consider

$$(35) \qquad u(x) = \inf_{\alpha \in \mathbf{A}} \int_0^\infty f(X_s,\alpha(s))e^{-\lambda s}ds$$

where

$$X_t = x + \int_0^t b(X_s,\alpha(s))ds \text{ for } t > 0.$$

Proposition I.4: Assume that (28), (29) and (30) hold with b, f independent of β and $\lambda > 0$. Then the function u given by (35) is uniformly continuous and is a strict viscosity supersolution of

$$(36) \qquad \lambda u(x) + \sup_{\alpha \in A} \{ - (b(x,\alpha),Du(x)) - f(x,\alpha)\} = 0 \text{ in } V$$

and u satisfies

$$(37) \qquad \lambda u(x)+\sup_{\alpha \in A}\{-(b(x,\alpha),p)-\varepsilon|b(x,\alpha)|-f(x,\alpha)\} \leq 0 \text{ for } x\in V,\ p\in D_\varepsilon^+u(x),\ \varepsilon > 0.$$

In particular, if $C_x = \{(b(x,\alpha), f(x,\alpha)) : \alpha \in A\}$ is convex for $x \in V$, then u is a strict viscosity solution of (36). Finally, if V satisfies (0), then u is the unique strict viscosity solution of (36) in UC(V).

We skip the proof, since it consists of straighforward variants of arguments given before (in particular, the verification of (37) is an easy adaptation of the method given in P. L. Lions [18]). Observe that the convexity of C_x leads to the identity

$$\sup_{\alpha\in A} \{-(b(x,\alpha),p) - \varepsilon|b(x,\alpha)| - f(x,\alpha)\} = \inf_{|q|<\varepsilon} \sup_{\alpha\in A} \{-(b(x,\alpha),p + q) - f(x,\alpha)\} .$$

I.3. An Example in a Nonsmooth Space.

The main point of this section is that if V does not have a differentiable norm (or, more generally, does not satisfy (0)), then the various notions of viscosity solution we have been using are not a suitable basis for a theory. This will be made clear by means of an explicit example in the space l^1 of summable sequences. Since l^1 does have the Radon-Nikodym property (it is separable and is the dual of c_0, and all separable dual spaces have this property), the difficulties to be exhibited are entirely associated with the lack of "smooth" functions on the space.

We will let φ denote the norm $\| \, \|_1$ of l^1

$$\varphi(x) = \|x\|_1 = \sum_{n=1}^{\infty} |x_n| \text{ for } x = \{x_n\}_{n\geq 1} \in l^1.$$

The dual space of l^1 is l^∞ and carries the dual norm

$$\|x\|_\infty = \sup_{n\geq 1} |p_n| \text{ for } p = \{p_n\}_{n\geq 1} \in l^\infty.$$

The example involves the Hamiltonian $H: l^1 \times l^\infty \to R$ given by

$$H(x,p) = -\|p\|_\infty + 1 - \varphi(x). \tag{38}$$

We claim that $u = \varphi$ is a strict viscosity solution of the equation

$$u + H(x,Du) = 0 \tag{39}$$

while $\hat{u} = (1/2)(\varphi + 1)$ is a strict viscosity solution up to 1/2 of

$$\hat{u} + H(x,D\hat{u}) \leq 0 \tag{40}$$

since $\hat{u} \leq \varphi$ holds on the sphere $\{\|x\|_1 = 1\}$ but not on the ball, the (boundary-value problem analogue) of Theorem I.1 does not hold.

To verify the claims, one first observes the relations

$$D^-\varphi(x) = \{p \in l^\infty: |p_n| \leq 1 \text{ and } p_n = \operatorname{sign} x_n \text{ if } x_n \neq 0\},$$

$$D_\varepsilon^-\varphi(x) = \{p \in l^\infty: \exists\, \hat{p} \in D^-\varphi(x) \text{ and } |p - \hat{p}|_\infty < \varepsilon\},$$

for all $\varepsilon > 0$ and $x \in l^1$, while

$$D_\varepsilon^+\varphi(x) = \emptyset \text{ if } 0 < \varepsilon \leq 1.$$

It is then straightforward to see that $u = \varphi$ is indeed a strict viscosity solution up to 1 of (39). It is completely trivial that $\hat{u}$ is a strict viscosity solution up to 1/2 of (40), since $D_\varepsilon^+\hat{u}(x) = \phi$ for $\varepsilon < 1/2$. (Observe that for λ, $\varepsilon > 0$, $D_{\lambda\varepsilon}^+\lambda u(x) = \lambda D_\varepsilon^+ u(x)$.) This completes the discussion of the example.

II. FURTHER REMARKS ON EXISTENCE RESULTS

II.1. Coercive Hamiltonians.

This section is devoted to variants on the existence results proved in Part II for (SP) and (CP). The distinguishing property of H which will be used in this development will be an assumption that H is large enough "at ∞" and this will allow us to relax (H3) to (H3w). For simplicity we will consider only bounded data and require V to be R^N, although we could proceed in the context of the strict viscosity solution theory. We will also use the blanket continuity assumption of Section 1.

For the stationary problem (SP) we have:

Theorem II.1: Let V be R^N and (0) hold. Assume (H2), (H3w), (H4) and that there are constants R_0, $C_0 > 0$ such that

(41) $|H(x,0,0)| \le C_0$ and $H(x,r,p) > C_0$ for $|p| > R_0$, $|r| \le C_0$ and $x \in V$.

Then there exists a unique bounded viscosity solution $u \in BUC(V)$ of (S). Moreover, u is Lipschitz continuous on V.

For the Cauchy problem (CP) we will assume that for a certain constant M given later

(42) $H(x,t,r,p) \to \infty$ as $|p| \to \infty$ uniformly for $(x,t,r) \in V \times [0,T] \times [-M,M]$.

(43) For each $R > 0$ there is a C_R such that

$$H(x,t,r,p) \le C_R$$

for $(x,t,r,p) \in V \times [0,T] \times [-M,M] \times B_R^*$.

(44) There is a modulus m such that

$$H(x,t,r,p) \le H(x,s,r,p) + m(t - s)$$

for $0 \le s \le t \le T$, $x \in V$, $|r| \le M$ and $p \in V^*$.

We have:

Theorem II.2: Let V be RN and (0) hold. Assume (H2), $(H3)_w$, (H4) and

$$|H(x,t,0,0)| \leq C_0.$$

for some constant C_0. Let $\varphi \in BUC(V)$,

$$C_1 > \sup_V |\varphi| \text{ and } M = C_0T + C_1.$$

Let (42), (43), (44) hold. Then there is a unique bounded viscosity solution of (CP) which is uniformly continuous on bounded sets and uniformly continuous in $x \in V$ uniformly in $t \in [0,T]$. Moreover, if φ is Lipschitz continuous on V, and $m(r) = Cr$ for some constant C, then u is LIpschitz continuous in x uniformly in $t \in [0,T]$.

Remarks II.1:

(i) Uniqueness is not a direct consequence of the uniqueness results in Parts I and II, but follows from these results combined with the existence of (approximate) Lipschitz continuous viscosity solutions .

(ii) It will be clear from the proof that we only need to assume (H2), (H3w) for $|r| \leq C_0$ in Theorem II.1 and for $|r| \leq M$ for Theorem II.2, since it is only in this range that the various constructions take place.

(iii) In the case when $V = \mathbb{R}^N$ the analogous results were obtained by G. Barles [4], extending previous results due to P. L. Lions [18], [19].

Sketch of Proof of Theorem II.1: We will make a number of reductions as in Part II ([10]) and take H(x,p) to be independent of r for simplicity as well. Observe that if we prove the existence of a bounded Lipschitz continuous viscosity solution of (SP) under the assumptions, then it will be unique among BUC(V) viscosity solutions of (SP) by the results of Part I ([9]). We begin the existence arguments by choosing $C_1 > C_0$ and truncating H at C_1; that is we replace H by

$$\tilde{H}(x,p) = \min(\max(H(x,p),-C_1),C_1). \tag{45}$$

Note that

$$\tilde{H}(x,p) > C_0 \text{ for } |p| > R_0,\ x \in V$$

because H satisfies the same condition. For $n = 1,2,\ldots$ and N as in (0) set

$$\chi_n(x) = (1 - nN(x))^+ \tag{46}$$

and then

(47) $$H_n(x,p) = \chi_n(x)\bar{H}(x,p) + (1 - \chi_n(x))C_0R_0^{-1}|p|.$$

Clearly

(48) $$H_n(x,p) > C_0 \text{ for } |p| > R_0,\ x \in V$$

and $H_n \in BUC(V \times B_R^*)$ for $R > 0$. To form the next approximation, we fix $R_1 > R_0$ and let T be the projection of V^* on $B_{R_1}^*$, that is

$$Tp = p \text{ if } |p| \leq R_1 \text{ and } Tp = (R_1/|p|)p \text{ if } |p| > R_1.$$

Then put

$$\bar{H}_n(x,p) = H_n(x,Tp).$$

Since $\bar{H}_n \in BUC(V \times V^*)$, the results of Part II (recall Remark I.1 (iii)) provide us with viscosity solutions $u_n \in BUC(V)$ of

$$u_n + \bar{H}_n(x,Du_n) = 0 \text{ in } V$$

and by the comparison results

$$|u_n| \leq \sup_V |\bar{H}_n(x,0)| \leq C_0.$$

Now one uses the equation, (48) and the definition of $\bar{H}_n$ to conclude that u_n is a viscosity solution of

(49) $$|Du_n| \leq R_0 \text{ on } V.$$

Using this we will show that

(50) $$u_n \text{ is Lipschitz with constant } R_0$$

(Lemma II.1 below). It follows from this that $D^{\mp}u_n(x) \subset B_{R_0}^*$ for all x and then that u_n is a viscosity solution of

(51) $$u_n + H_n(x,Du_n) = 0 \text{ on } V.$$

Since $H_n \to H$ uniformly on $B_R \times V^*$ for all R, we may invoke the various assumptions, (50) and (51) to deduce from the convergence theorem in Part II ([10]) (restated to use the uniform Lipschitz continuity (50) and (H3w) rather than (H3)) to conclude that u_n converges uniformly on bounded sets of V to a viscosity solution u of (SP) which satisfies (49) and (50).

It remains to establish:

Lemma II.1: Let V be RN and satisfy (0), $R > 0$ and $u \in BUC(B_R)$ for all $R > 0$.

(i) Let $w \in V$, $f \in BUC(B_R)$ for all $R > 0$ and u be a viscosity solution of

(52) $$(Du,w) \leq f(x) \text{ in } V.$$

Then if $s > 0$

(53) $$u(x + sw) - u(x) \leq \int_0^s f(x + \tau w)d\tau \text{ in } V .$$

(ii) If $L > 0$ and u is a viscosity subsolution of

(54) $$|Du| \leq L \text{ in } V$$

then

(55) $$|u(x) - u(y)| \leq L|x - y| \text{ for } x, y \in V.$$

Proof of Lemma II.1: The assertion (ii) is a consequence of (i) since (54) implies that u is a viscosity solution of $(Du,w) \leq C$ in V for all $w \in B_1$. There are several arguments which can be used to prove (i). One way is to consider $u(x)$ as a time-independent subsolution of

(56) $$v_t + (Dv,w) = f \text{ in } V \times [0,\infty)$$

which satisfies the intial condition

(57) $$v(0,x) = u(x) \text{ in } V.$$

Then set $v(x,t) = u(x - wt) + \int_0^t f(x - ws)ds$ and check that v is a viscosity solution of (56) satisfying (57). Since $v \in BUC(B_R \times [0,T])$ for $R,T > 0$, the proofs of the comparison results of Part I ([9]) adapt to establish $u \leq v$. (The requirement that u or v be uniformly continuous in x uniformly in t in the results of Part I can be replaced by using finite speed of propogation - the finite dimensional results of [7] extend to infinite dimensions - or by the general results of [12].)

Sketch of Proof of Theorem II.2: The asserted uniqueness follows from the approximations constructed during the existence arguments. To prove the existence we will use the method of G. Barles [4]. Let $\varepsilon > 0$ and consider the function

$$H_\varepsilon(x,t,r,p) = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} \inf_{\tau\leq\lambda\leq t} H(x,\lambda,r,p)\, d\tau$$

where we put $H(x,t,r,p) = H(x,0,r,p)$ for $t \leq 0$. In view of (44) we have the following relations:

$$\text{(i)}\quad H(x,t,r,p) \geq H_\varepsilon(x,t,r,p) \geq H(x,t,r,p) - \omega(\varepsilon), \tag{58}$$

$$\text{(ii)}\quad H_\varepsilon(x,t,r,p) - H_\varepsilon(x,s,t,p) \leq \frac{\omega(T)}{\varepsilon}(t - s) \text{ for } 0 \leq s \leq t \leq T.$$

Indeed, to prove (58) observe that if $t \geq s + \varepsilon$, then the difference is bounded by $\omega(t - s)$, while if $t \leq s + \varepsilon$ we may write (surpressing the variables x, r, p)

$$\begin{aligned} H_\varepsilon(t) - H_\varepsilon(s) &= \frac{1}{\varepsilon}\int_s^t \Big(\inf_{\tau\leq\lambda\leq t} H(\lambda) - \inf_{\tau-\varepsilon\leq\lambda\leq s} H(\lambda)\Big)d\tau + \\ &+ \frac{1}{\varepsilon}\int_{t-\varepsilon}^s \Big(\inf_{\tau\leq\lambda\leq t} H(\lambda) - \inf_{\tau\leq\lambda\leq s} H(\lambda)\Big)d\tau \end{aligned} \tag{59}$$

and observe that the second integral is nonpositive while the first one is bounded by $\omega(\varepsilon)(t - s)/\varepsilon$.

Next we claim that for each Lipschitz continuous approximation φ_ε of φ (e.g., the inf convolution $\varphi_\varepsilon(x) = \inf_{y\in V}(\varphi(y) + |x - y|/\varepsilon)$ there is a bounded viscosity solution u_ε of

$$u_{\varepsilon t} + H_\varepsilon(x,t,Du_\varepsilon) = 0 \text{ on } V \times (0,T],\ u_\varepsilon(x,0) = \varphi_\varepsilon(x) \text{ on } V$$

which is uniformly continuous on bounded sets and is Lipschitz continuous in $x \in V$ uniformly in $t \in [0,T]$. Once this claim is established, then it is easy to show that such a solution is unique, satisfies $|u_\varepsilon| \leq M$ on $V \times [0,T]$ and, using the comparison results of Part I ([9]),

$$\sup_{V\times[0,T]} |u_\varepsilon - u_\eta| \leq (\omega(\varepsilon) + \omega(\eta))T + (\varepsilon + \eta).$$

Hence u_ε converges uniformly to some u with the properties asserted in the existence theorem. Moreover, the uniqueness results follow upon comparison of another assumed solution with u_ε, which is Lipschitz continuous in x.

The existence of u_ε is obtained by using reductions of the sort outlined in the proof of Theorem II.1 in conjunction with a priori estimates we now describe. First of all, comparison shows that u_ε is bounded by M. Next, one shows that there is a constant C_ε such that

(60) $$u_\varepsilon(x,t + h) \geq u_\varepsilon(x,t) - C_\varepsilon h \text{ for } h > 0.$$

This follows from (59) and comparison results - observe that the inequality will hold at $t = 0$ since $\varphi_\varepsilon(x) - ct$ is a viscosity subsolution for large C (use (43)). Then one uses the equation satisfied by u_ε and (42) to obtain bounds on $|Du_\varepsilon|$ uniform in t. Using these estimates and the methods by which they are obtained, the existence of u_ε is established using reductions similar to the ones employed in the proof of Theorem I.1.

II.2. Remarks on Faedo - Galerkin Approximations.

The main thrust of this section is that Galerkin procedures, naively posed, dramatically fail to approximate the solutions provided by the theory we are developing. The search for of convergent finite dimensional approximation procedures remains an interesting open area in the subject.

We will restrict our attention in this section to the case in which V is a separable (real) Hilbert space with which we identify the dual V^*; $V^* = V$. Thus (,) will be the inner-product of V. Let us consider a Hamiltoninan $H(p)$ which is a function of $p \in V$ alone and is uniformly continuous on bounded sets. It follows from the results of Parts I and II that for each $\varphi \in UC(V)$ there is a unique viscosity solution of the problem

(61) $$u_t + H(Du) = 0 \text{ in } V \times (0,\infty),\ u(x,0) = \varphi(x) \text{ in } V$$

which is continuous on bounded sets and uniformly continuous on V uniformly in t. Let $V_1 \subset V_2 \subset \cdot \cdot \subset V_n \subset \cdot$ be an increasing sequence of finite dimensional subspaces of V whose union is dense in V. Denoting the restrictions of H and φ to V_n by H_n and φ_n, it is natural to hope that the solution u_n of

(62) $$u_{nt} + H_n(Du_n) = 0 \text{ in } V_n \times [0,\infty),\ u_n(x,0) = \varphi_n(x) \text{ on } V_n$$

will converge to the solution u of (62). Indeed, one might hope to base the existence theory on this standard Faedo - Galerkin method.

However, we claim that, in general, this method does not converge. In fact, we next present an explicit example in which the solutions u_n of (62) converge to a function $\bar{u}$ which is not the solution of (61). We console ourselves with the remark that the problem (61) should be regarded as a problem in BUC(V) and not in V, so it is too not so surprising that simply approximating V does not succeed here. The possible utility of further compactness conditions in this direction remains to be investigated. One last comment before the example: There are very many possible choices of V_n, H_n, φ_n which one might make in any particular problem. The example below shows they may be choosen badly. The possibility remains that they could be choosen well.

Example II.1: Let V be the Hilbert space of square summable doubly infinite sequences $x = \{x_n\}_{n \neq 0}$ equipped with the norm

$$|x|_2 = \Big(\sum_{n \neq 0} |x_n|^2 \Big)^{1/2}$$

Put

$$H(p) = \sup_{y \in E} (p,y) \text{ where } E = \{p \in V: |p| \leq 1 \text{ and } p_n = p_{-n} \text{ for } n \neq 0\}$$

where we have used the letter E because its elements are even. Clearly H is convex and Lipschitz continuous on V. It follows that (60) has a "finite speed of propagation" and we may consider the problem for arbitrary $\varphi: V \to R$ which is uniformly continuous on bounded sets. We choose

$$\varphi(x) = \sum_{j \geq 1} (x_j^2 - x_{-j}^2)$$

Next we set

$$V_n = \{x \in V: x_j = 0 \text{ for } j \geq n+1 \text{ or } j \leq -n-2\}.$$

The Lax - Oleink formula for the solution of problems like (61) (see [18] for the finite-dimensional case - the same formula holds here) presents the unique viscosity solutions of (61), (62) in the form

(63) $$u(x,t) = \inf \{\varphi(y): y \in x + tE\} \text{ for } (x,t) \in V \times [0,\infty)$$

and

(64) $$u_n(x,t) = \inf \{\varphi(y): y \in x + tP_nE\} \text{ for } (x,t) \in V_n \times [0,\infty)$$

where P_n is the orthogonal projection onto V_n. Now, fixing $x \in V_{n_0}$ and let $n \to \infty$ we easily compute that

$$u_n(x,t) = \inf \Big\{ \sum_{j=1}^{n_0} (x_j + tk_j)^2 - \sum_{j=1}^{n_0} (x_{-j} + tk_j)^2 + t^2k_{n_0+1}^2 - (x_{-n_0-1} + tk_{n_0+1})^2$$

$$- t^2k_{n+1}^2 : \sum_{j=1}^{n_0+1} k_j^2 + k_{n+1}^2 \leq 1/2 \Big\}$$

$$= \varphi(x) + \inf \Big\{ -2t \sum_{j=1}^{n_0} k_j |x_j - x_{-j}| - 2tk_{n_0+1} |x_{-n_0-1}| - t^2 k_{n+1}^2 :$$

$$\sum_{j=1}^{n_0+1} k_j^2 + k_{n+1}^2 < \frac{1}{2}, \ k_j > 0 \text{ for } 1 < j < n_0 + 1 \Big\}$$

or

$$u_n(x,t) = \begin{cases} \varphi(x) - \frac{t^2}{2} - \sum_{j=1}^{n_0+1} X_j^2 & \text{if } \sum_{j=1}^{n_0+1} X_j^2 < \frac{t^2}{2} \\ \varphi(x) - t\sqrt{2}\left(\sum_{j=1}^{n_0+1} X_j^2\right)^{1/2} & \text{if } \sum_{j=1}^{n_0+1} X_j^2 > \frac{t^2}{2} \end{cases}$$

with $X_j = |x_j - x_{-j}|$ (recall that for $j = n_0 + 1$, $x_{n_0+1} = 0$). Therefore, putting $X_j = |x_j - x_{-j}|$, u_n converges uniformly on each subspace V_{n_0} to the function $\hat{u}$ given by

$$\hat{u}(x,t) = \begin{cases} \varphi(x) - \frac{t^2}{2} - \sum_{j=1}^{\infty} X_j^2 & \text{if } \sum_{j=1}^{\infty} X_j^2 < \frac{t^2}{2} \\ \varphi(x) - \sqrt{2}t\left(\sum_{j=1}^{\infty} X_j^2\right)^{1/2} & \text{if } \sum_{j=1}^{\infty} X_j^2 > \frac{t^2}{2}. \end{cases}$$

The function $\hat{u}$ is Lipschitz continuous on $B_R \times [0,T]$ for all R, $T > 0$, C^1 on the open sets $\sum_{j=1}^{\infty} X_j^2 < \frac{t^2}{2}$ and $\sum_{j=1}^{\infty} X_j^2 > \frac{t^2}{2}$ and it satisfies (61) on the latter of these sets. However, it is not a solution of (61) since

$$\hat{u}_t + H(D\hat{u}) = -t + \sqrt{2}\left(\sum_{j=1}^{\infty} X_j^2\right)^{1/2} < 0 \text{ on } \sum_{j=1}^{\infty} X_j^2 < \frac{t^2}{2}.$$

<u>Example II.2</u>: Let H be continuous and convex on V. We assume that for all n large enough

(65) $$(H|_{V_n})^* = H^*|_{V_n}$$

where * denotes the operation of taking the convex conjugate. This is equivalent to

(65)' $$\sup_{y \in V_n} \{(p,y) - H(y)\} = \sup_{y \in V} \{(p,y) - H(y)\} \text{ for } p \in V_n .$$

Observe that $H(p) = \Phi(|p|)$ is a particular case for which (65) holds. In this case the unique solution of (62) is given by the Lax-Oleinik formula [18] as

$$u_n(x,t) = \inf \{ \varphi(y) + tH^*((x - y)/t) : y \in V_n \}.$$

It is clear from this that $u_n(x,t) > u(x,t)$ for all $x \in V$ and $t > 0$, but we will see the convergence is not uniform on balls in V_n. Indeed, take

$$H(p) = \tfrac{1}{2}|p|^2, \; V = l^2 = \{x = \{x_n\}_{n>1} : \sum_{n=1}^{\infty} x_j^2 < \infty \}$$

$V_n = \{x \in V: \; x_j = 0 \text{ if } j > n + 1 \}$ and

$$\varphi(x) = \frac{1}{2} \sum_{k=1}^{\infty} \{ x_{2k-1}^2 + x_{2k}^2 + 2\alpha x_{2k-1}x_{2k} \}$$

on some large ball, where $|\alpha| < 1$. Using finite speed of propagation, we may assume this form of φ globally and the results will be valid locally. Let e_n be the coordinate vector with 1 in the n^{th}-slot and zero elsewhere. One computes that

$$u_n(e_n,t) = \frac{1}{2} \frac{1}{1 + t}$$

while

$$u(e_n,t) = \tfrac{1}{2}(1 + 3t + 3t^2 + t^3(1 - \alpha^2)^2 - \alpha^2 t - 3\alpha^2 t^2)((1 + t)^2 - (t\alpha)^2)^{-2}$$

and if $\alpha \neq 0$ these two quantities differ for all but a few values of t.

II.3. Some Hamilton-Jacobi Equations Arising in the Optimal Control of Evolution Equations.

In [1], [2], V. Barbu and G. Da Prato study optimal control problems of general abstract evolution equations in a Hilbert space V, which we identify with its dual as in the preceeding section. By the usual dynamic programming argument, they were led to a class of Hamilton-Jacobi equations, a particular case of which is has the following form:

$$u_t + (Ax,Du) + F(t,Du) = 0 \text{ in } V \times (0,T] \tag{66}$$

where $-A$ is the infinitesimal generator of a strongly continuous semigroup e^{-tA}, $t > 0$, of bounded linear transformations on V and $F(t,p)$ is continuous on $[0,T] \times V$ and there is a local modulus σ such that (at least) for $R > 0$

$$|F(t,p) - F(t,q)| \leq \sigma(|p - q|,R) \text{ for } t \in [0,T] \text{ and } p, q \in B_R. \tag{67}$$

Here we show how a simple device tranforms the problem (66) into one which (almost) fits into the classes studied in Parts I and II. We are able to solve the resulting equation and we check that this transformation is consistent with the value functions of some optimal control problems. However, the success of this device will depend essentially on the assumption that F in (66) does not depend on x and we therefore exclude more general equations studied by Barbu and Da Prato in [1], [2].

The device we use to treat (66) is the simple (and purely formal for now) change of unknown function given by

$$u(x,t) = v(e^{-tA}x,t). \tag{68}$$

Calculating formally, one sees that if u solves (66) then v should solve

$$v_t + H(t,Dv) = 0 \text{ in } V \times (0,T] \tag{69}$$

where

$$H(t,p) = F(t,e^{-tA^*}p) \tag{70}$$

and e^{-tA^*} is the adjoint semigroup (generated by the adjoint $-A^*$ of $-A$). If A is bounded and u and v are related by (68), then u is a viscosity solution of (66) if and only if v is a viscosity solution of (69), so the correspondence is perfectly sensible. In the general case, we will call u a viscosity solution of (66) if it is given by (68) where v is a viscosity solution of (69). To support the appropriateness of this procedure we will subsequently show that when u is given by (68) and v is the viscosity solution of (69) which we construct, then u is indeed the value function of the associated optimal control problem.

We begin the discussion of existence and uniqueness for (69). First, H as given by (70) is clearly continuous on $[0,T] \times V$ and satisfies (67) (that is, H is uniformly continuous in p on bounded sets). The results of Parts I and II ([9], [10]) do not apply directly since in those works we assumed that H was uniformly continuous in t on bounded sets. This is the case above if A is bounded, but not otherwise. However, an inspection of the proofs in [9] shows that this continuity property with respect to t was not needed - it is enough to have continuity and (66) (Lemma 1 of [10] still holds and then the uniqueness proof runs as usual). By contrast, the continuity in t uniformly on bounded sets was used in a significant way in the existence program of [10] in one step - the verification that certain value functions for differential games were viscosity solutions of the associated Hamilton-Jacobi equations. It does not seem easy to relax the assumptions to the degree needed here in any general way. However, the explicit structure we are dealing with allows us to proceed in this case. Strengthening (67) to

(71) F is uniformly continuous on bounded subsets of $[0,T] \times V$

the existence program of [10] now succeeds for (69), even though we do not improve the continuity of H in t! The reason for this is that the structure of H, which we regard as a special case of the general form $H(x,t,r,p) = F(x,t,r,S(t)p)$ where F is uniformly continuous on bounded sets, satisfies the usual existence and uniqueness conditions and $t \to S(t)$ is a strongly continuous mapping into the bounded linear self-maps of V. The reason that this structure allows one to avoid the difficulty mentioned above is that (in the notation of [10]) the quantity $(S(t)p,\xi(q(s)))$ is now continuous in t uniformly for $q(s) \in Q$. With these remarks, we deduce from the proofs in Parts I - II the following result:

Proposition II.1: Let (67) hold and $\varphi \in UC(V)$.

(i) There is at most one viscosity solution v of (69) which is uniformly continuous on bounded sets, uniformly continuous in x uniformly in $t \in [0,T]$ and satisfies $v(x,0) = \varphi(x)$.

(ii) If also (71) holds, then there exists a viscosity solution $v \in UC(V \times [0,T])$ of (69) satisfying $v(x,0) = \varphi(x)$ on V. Moreover, if φ is bounded, then $v \in BUC(V \times [0,T])$ while if φ is Lipschitz on V, then v is Lipschitz on $V \times [0,T]$.

We conclude by showing that the above transformation is compatible with optimal control problems. For simplicity, F will be taken to be a t-independent convex function of $p \in V$. F^* will denote the conjugate convex function of F.

To formulate a control problem associated with (66) consider the state equation

$$\frac{d}{dt} X_t + AX_t + \alpha_t = 0 \text{ for } t > 0,\ X_0 = x$$

where $x \in V$ and the control α lies in $L^1(0,T;V)$. Then X_t is given by

$$X_t = e^{-tA}x - \int_0^t e^{-(t-s)A}\alpha_s ds.$$

We may define the following value function

$$u(x,t) = \inf_{\alpha} \left\{ \int_0^t F^*(\alpha_s)ds + \varphi(X_t)\right\}. \tag{72}$$

The notations and assumptions above being in force, we have:

Proposition II.2: Let v be the solution of (69) given by Proposition II.1 and u be given by (72). Then

$$u(x,t) = v(e^{-tA}x,t) \text{ for } x \in V,\ t \in [0,T]. \tag{73}$$

Proof: Let $t > 0$ and $x, y \in V$ and $y = e^{-tA}x$. Set $w(y,t) = u(x,t)$ so that w is given by the formula

(74) $$w(y,t) = \inf_{\alpha} \left\{ \int_0^t F^*(\alpha_s)ds + \varphi(y - \int_0^t e^{-(t-s)A}\alpha_s ds)\right\}$$

which reveals that $w(y,t)$ does not depend on the choice of x (if it is not unique). That is, w is well-defined on $\{(y,t): t \in [0,T] \text{ and } y \in R(e^{-tA})\}$. If we show that $v = w$ on this set, the claim is proved.

By the usual dynamic programming arguments, (74) yields

(75) $$w(x,t) = \inf_{\alpha} \left\{ \int_0^{t-\tau} F^*(\alpha_s)ds + w(x - \int_0^{t-\tau} e^{-(t-s)A}\alpha_s ds, \tau) \right\}.$$

for $t > \tau > 0$. We will now assume that φ is Lipschitz since a density argument shows that it suffices to prove that $w = v$ in this situation. Recalling from convex analysis that F^* satisfies

(76) $$D(F^*) \neq \emptyset \text{ and } F^*(q)/|q| \to \infty \text{ as } |q| \to \infty$$

we deduce easily that w is finite on $V \times [0,T]$. It is obvious from the formula that $w(y,t)$ is Lipschitz continuous in y uniformly in t since φ is Lipschitz. We next claim that w is Lipschitz in t. Indeed, let $h > 0$ and let $t > h$. Choose $\alpha_t \equiv \alpha \in D(F^*)$ (a constant control) and then deduce from (75) with $\tau = t - h$ and the inf estimated above by value for this constant control that

(77) $$\begin{aligned} w(x,t) &\leq hF^*(\alpha) + w(x - \int_0^h e^{-(t-s)A}\alpha ds, t - h) \leq \\ &\leq hF^*(\alpha) + w(x,t-h) + CMe^{\omega h}|\alpha|h \end{aligned}$$

where C is a Lipschitz constant for w in x and M and ω are constants such that

$$|e^{-tA}| \leq Me^{\omega t}.$$

On the other hand

$$w(x,t) \geq \inf_{\alpha} \left\{ \int_0^h F^*(\alpha_s)ds - CMe^{\omega h}\int_0^h |\alpha_s|ds \right\} + w(x,t-h)$$

and using (76) we find that for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$|x| \leq \varepsilon F^*(x) + C_\varepsilon.$$

Using this above we see by choosing ε sufficiently small we have

$$w(x,t) \geq -Kh + w(x,t-h)$$

and it follows then that w is Lipschitz continuous of $V \times [0,T]$. These considerations also show that we may restrict the infimum in (75) to controls α_t such that

$$(78)\qquad \int_0^{t-\tau} \left(|F^*(\alpha_s)| + |\alpha_s|\right)ds \leq K(t-\tau) \text{ for } 0 \leq \tau \leq t \leq T.$$

Finally we claim that w is a viscosity solution of (69). This will complete the proof, for the continuity properties of w established above and the uniqueness result of Proposition II.1 then yield that $v = w$. The proof that w is a viscosity subsolution of (69) follows the usual lines ([18]) and is quite easy since for every $\alpha \in D(F^*)$ we have (77) and, moreover,

$$\frac{1}{h}\left|\int_0^h e^{-(t-s)A}\alpha ds - he^{-tA}\alpha\right| \to 0 \text{ as } h \to 0+$$

We then obtain for any such α

$$w_t + (e^{-tA}\alpha, Dw) - F^*(\alpha) \leq 0 \text{ in } V \times (0,\infty)$$

in the viscosity sense. Taking the supremum over $\alpha \in D(F^*)$ we conclude that w is a viscosity subsolution of (69).

Finally, to prove that w is a viscosity supersolution, let $(p,\gamma) \in D^-w(z,s)$ where $s > 0$, $\gamma \in \mathbb{R}$ and $p \in V$. That is,

$$w(y,t) \geq w(z,s) + (p,y-z) + \gamma(t-s) + o(|y-z| + |t-s|).$$

Using (74) (with $t = s$, $t' = s-h$, $h > 0$ small) and (78) we deduce that

$$\lambda + \sup{}' \left\{\frac{1}{h}\left(p, \int_0^h e^{-A(s-\tau)}\alpha_\tau d\tau\right) - \frac{1}{h}\int_0^h F^*(\alpha_s)ds\right\} \geq \varepsilon(h)$$

where $\varepsilon(h) \to 0$ as $h \downarrow 0$ and "sup'" refers to taking the supremum over controls α_t satisfying (78) (with $t - t' = h$). Next we observe that for such α_τ

$$\left|\frac{1}{h}\left(p, \int_0^h e^{-A(s-\tau)}\alpha_\tau d\tau\right) - \frac{1}{h}\left(e^{-A^*s}p, \int_0^h \alpha_\tau d\tau\right)\right| \leq \frac{1}{h}\int_0^h \left|e^{-A^*(s-\tau)}p - e^{-A^*s}p\right| |\alpha_\tau| d\tau$$

$$\leq C \sup_{0\leq s\leq h} \left|e^{-A^*(s-\tau)}p - e^{A^*s}p\right|$$

and the right hand side tends to 0 as $h \downarrow 0$. Finally we obtain

$$\liminf_{h\downarrow 0} \lambda + \sup{}'\left\{ e^{-A^* s}p, \frac{1}{h}\int_0^h \alpha_s ds\right) - \frac{1}{h}\int_0^h F^*(\alpha_\tau)d\tau \left.\vphantom{\int}\right\} \geq 0$$

which obviously implies

$$\limsup_{h\downarrow 0} \lambda + \sup_{\alpha} \left\{ e^{-A^* s}p, \frac{1}{h}\int_0^h \alpha_\tau d\tau\right) - \frac{1}{h}\int_0^h F^*(\alpha_\tau)d\tau \right\} \geq 0 .$$

Now we are done since the left-hand side is nothing but $\lambda + F(e^{-A^* s}p)$.

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This paper is concerned with a number of topics in the theory of viscosity solutions of Hamilton-Jacobi equations in infinite dimensional spaces begun in Parts I and II of this series. The development of the theory in the generality in which the "space" or state variable lies in an infinite dimensional space is partly motivated by the hope of eventual applications to the

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theory of control of partial differential equations or control under partial observation. Among the results presented are: The existence and uniqueness theory previously discussed in spaces with the Radon-Nikodym property is extended beyond this class; examples are given which show that Galerkin approximation arguments in their naive forms cannot be made the basis of an existence theory; some equations with "unbounded terms" of the sort that arise in control of pde's are treated by means of a change of variables reducing the problem to the previously studied cases.

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